On Connectivity of Discretized 2D Explicit Curve

Fumiki Sekiya and Akihiro Sugimoto

Abstract Preserving connectivity is an important property commonly required for object discretization. Connectivity of a discretized object differs depending on how to discretize its original object. The morphological discretization is known to be capable of controlling the connectivity of a discretized object, by selecting an appropriate structuring element. The analytical approximation, which approximates the morphological discretization by a finite number of inequalities, on the other hand, is recently introduced to reduce the computational cost required for the morphological discretization. However, whether this approximate discretization has the same connectivity that the morphological discretization has is yet to be investigated. In this paper, we study the connectivity relationship between the morphological discretization and the analytical approximation, focusing on 2D explicit curves. We show that they guarantee the same connectivity for 2D explicit curves.

Keywords Discretization · Explicit curve · Connectivity · Morphological discretization · Structuring element · Analytical approximation

1 Introduction

An object such as a curve or line is continuous in the real world while in the computer it is discretized to be stored and manipulated. We therefore need a discrete representation of a given object, which differs depending on how to discretize it. An important property commonly required for object discretization is to preserve the connectivity of an original object. In this paper, we consider discretization of an

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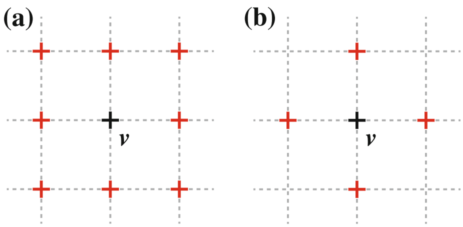

Fig. 1 k-adjacent points of an integer point $\boldsymbol{v}$. **a** $k = 0$. **b** $k = 1$

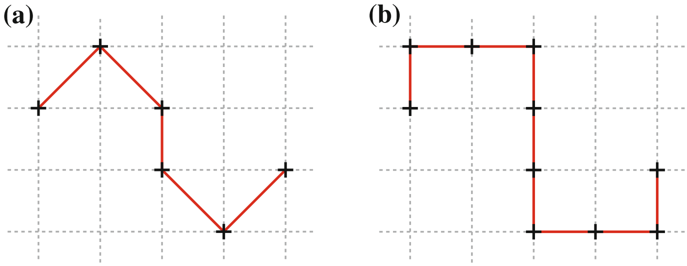

Fig. 2 k-connected sets in $\mathbb{Z}^2$. *Red line segments* indicate k-adjacency relation in the set. **a** $k = 0$. **b** $k = 1$

explicit continuous curve in 2D, i.e., a continuous function in the form of $y = f(x)$ in the xy-plane, primarily focusing on the connectivity of discretized curves.

Two integer points $\boldsymbol{v}, \boldsymbol{w}$ $(\boldsymbol{v} \neq \boldsymbol{w}) \in \mathbb{Z}^2$ are said to be 0-*adjacent* if $\|\boldsymbol{v} - \boldsymbol{w}\|_\infty \leq 1$, and 1-*adjacent* if $\|\boldsymbol{v} - \boldsymbol{w}\|_1 \leq 1$ (Fig. 1). Note that $\|(x, y)\|_\infty = \max(|x|, |y|)$ and $\|(x, y)\|_1 = |x| + |y|$. A set of integer points $D \subset \mathbb{Z}^2$ is said to be k-*connected* for $k \in \{0, 1\}$, if for any two points $\boldsymbol{v}, \boldsymbol{w}$ $(\boldsymbol{v} \neq \boldsymbol{w}) \in D$ there exists a sequence of integer points in D connecting $\boldsymbol{v}$ and $\boldsymbol{w}$, such that any two consecutive points in the sequence are k-adjacent. Figure 2 shows k-connected sets for $k = 0, 1$. We remark that if D is 1-connected then it is also 0-connected.

The discretization most commonly used is the *morphological discretization* [11–14]. In this approach, for a continuous curve, its discretized curve is defined as a set of the integer points, whose Minkowski additions with a so-called structuring element intersect with the original curve. Some classical discretizations, such as the supercover discretization [9] or the grid-intersection discretization [15], can be seen as particular cases of the morphological discretization. The morphological discretization can control the connectivity in the discrete space of a discretized curve by selecting an appropriate structuring element [6–8, 21–23].

How to discretize a curve and how to compute its discretized one are different issues. The computational cost required for the morphological discretization is expensive. To overcome this drawback, representing a discretized curve by a finite set of Diophantine inequalities (from which we choose only integer points) was introduced in [19], where a discrete 2D straight line is defined by two inequalities.

Such a representation, called the analytical representation, has been developed for more complicated discrete curves in subsequent researches [1–5, 10, 22, 23]. A discretized curve with an analytical representation is straightforwardly computed at low cost, just by evaluating inequalities for each integer point. This property is useful also for curve fitting problems [16–18, 20, 24, 25]. To reduce the computational cost further, an approximation of the analytical representation, called the *analytical approximation*, was recently introduced in [22], where only vertices of the employed structuring element are evaluated to have the system of Diophantine inequalities. This approximation is capable of handling even further complicated (and implicit) curves/surfaces in any dimensions. However, whether the analytical approximation has the same connectivity that the original morphological discretization has is yet to be investigated.

In this paper, we study the relationship on the connectivity between discretized 2D explicit curves, by the morphological discretization and by the analytical approximation. We show that the analytical approximation has the same connectivity that the morphological discretization has for 2D explicit curves.

2 Morphological Discretization and Analytical Approximation

In this section, we first introduce the definition of the morphological discretization, with two structuring elements guaranteeing 1-connectivity or 0-connectivity (Sect. 2.1). We then give the analytical approximation of the morphological discretization of a 2D explicit curve with these structuring elements, based on the approach introduced in [22] (Sect. 2.2).

2.1 Morphological Discretization

The *morphological discretization* (see [11–14]) of a curve $E \subset \mathbb{R}^2$, with a *structuring element* $S \subset \mathbb{R}^2$, is defined by

$$D_S(E) = (E \oplus \check{S}) \cap \mathbb{Z}^2, \tag{1}$$

where $\check{S} = \{-\boldsymbol{s} : \boldsymbol{s} \in S\}$. $\oplus$ denotes the Minkowski addition ($E \oplus \check{S} = \{\boldsymbol{e} + \check{\boldsymbol{s}} : \boldsymbol{e} \in E, \check{\boldsymbol{s}} \in \check{S}\}$). (1) can be also written as

$$D_S(E) = \left\{\boldsymbol{v} \in \mathbb{Z}^2 : (\{\boldsymbol{v}\} \oplus S) \cap E \neq \emptyset\right\}. \tag{2}$$

Figure 3 illustrates the two different interpretations of $D_S(E)$ in (1) and (2).

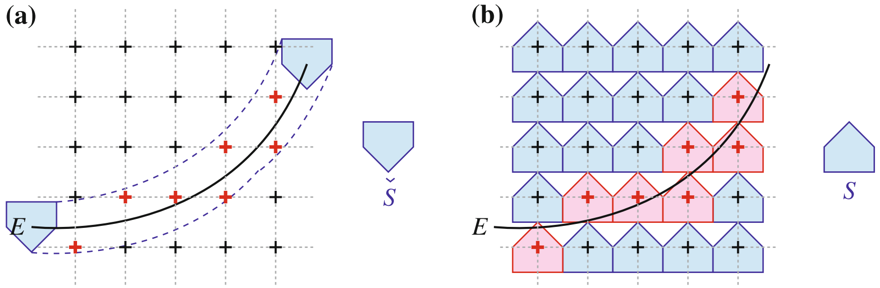

Fig. 3 Two different interpretations of morphological discretization $D_S(E)$. *Red points* depict $D_S(E)$. **a** $D_S(E)$ in (1). **b** $D_S(E)$ in (2)

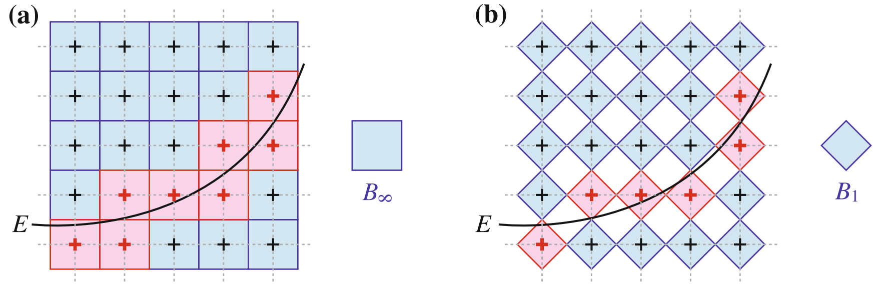

Fig. 4 Morphological discretizations using structuring elements B_∞ and B_1. **a** is 1-connected while **b** is 0-connected. **a** $D_{B_\infty}(E)$. **b** $D_{B_1}(E)$

Using different structuring elements for the same curve results in different discretizations, and in particular, different connectivities (see Fig. 4 for example). How to select an appropriate structuring element is therefore important. In this paper, we focus on two structuring elements, each of which induces 1-connectivity or 0-connectivity. They are defined by

$$B_\infty = \left\{ \boldsymbol{p} \in \mathbb{R}^2 : \|\boldsymbol{p}\|_\infty \leq \frac{1}{2} \right\},$$
$$B_1 = \left\{ \boldsymbol{p} \in \mathbb{R}^2 : \|\boldsymbol{p}\|_1 \leq \frac{1}{2} \right\}.$$

The morphological discretization with B_∞, i.e., $D_{B_\infty}(E)$, is equivalent to the supercover discretization of E, which is known to be 1-connected if E is connected in $\mathbb{R}^2$ [21] (Fig. 4a). $D_{B_1}(E)$, on the other hand, is 0-connected for connected E (Fig. 4b), which has yet to be reported to the best of our knowledge; here we give its proof.

Theorem 1 *$D_{B_1}(E)$ is 0-connected for connected $E \subset \mathbb{R}^2$, as long as it has at least two different integer points.*

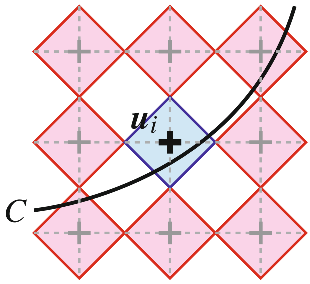

Fig. 5 Illustration for proof of Theorem 1. *Blue region* depicts $\{\boldsymbol{u}_i\} \oplus B_1$, while *red region* depicts $A_0(\boldsymbol{u}_i) \oplus B_1$

Proof Let $\boldsymbol{s}, \boldsymbol{t}$ be any two different integer points in $D_{B_1}(E)$. We show that there exists a sequence of integer points from $\boldsymbol{s}$ to $\boldsymbol{t}$ in $D_{B_1}(E)$, such that any two consecutive points in the sequence are 0-adjacent. We call such a sequence a 0-*path* from $\boldsymbol{s}$ to $\boldsymbol{t}$ in $D_{B_1}(E)$. We denote by $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^2$ intersection points of E respectively with $\{\boldsymbol{s}\} \oplus B_1$ and $\{\boldsymbol{t}\} \oplus B_1$. Then, since E is connected, there exists a segment $C \subset E$ whose end points are $\boldsymbol{p}$ and $\boldsymbol{q}$. We now consider the unique path along C from $\boldsymbol{p}$ to $\boldsymbol{q}$, with collecting in the path integer points $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$, whose Minkowski additions with B_1 (i.e., $\{\boldsymbol{u}_i\} \oplus B_1, i = 1, \ldots, n$) intersect with C. This process generates a 0-path from $\boldsymbol{s}$ to $\boldsymbol{t}$ in $D_{B_1}(E)$, which is proven as follows. First, it is obvious that $\boldsymbol{u}_1 = \boldsymbol{s}$ and $\boldsymbol{u}_n = \boldsymbol{t}$. Next, $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\} \subset D_{B_1}(E)$ because $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\} = D_{B_1}(C)$ and $C \subset E$. Finally, we show that $\boldsymbol{u}_{i+1}$ is a 0-adjacent point of $\boldsymbol{u}_i$ for $i = 1, \ldots, n-1$. We denote by $A_0(\boldsymbol{v})$ the set of the 0-adjacent points of $\boldsymbol{v} \in \mathbb{Z}^2$ (see Fig. 1a). Any point in $\{\boldsymbol{u}_i\} \oplus B_1$ is then either contained in $A_0(\boldsymbol{u}_i) \oplus B_1$ or enclosed by it as in Fig. 5. Therefore, the path along C from an intersection point with $\{\boldsymbol{u}_i\} \oplus B_1$ toward the terminal point $\boldsymbol{q}$, has to cross $A_0(\boldsymbol{u}_i) \oplus B_1$ before reaching the "outside" of it. This indicates that $\boldsymbol{u}_{i+1} \in A_0(\boldsymbol{u}_i)$. There exists a 0-path from $\boldsymbol{s}$ to $\boldsymbol{t}$, accordingly. □

2.2 Analytical Approximation

Computing $D_S(E)$ for a given $E \subset \mathbb{R}^2$ with $S = B_\infty, B_1$ requires evaluating for any $\boldsymbol{v} \in \mathbb{Z}^2$ whether or not $\{\boldsymbol{v}\} \oplus S$ intersects with E. This is computationally expensive. When E is an explicit curve, i.e., in the form of $y = f(x)$, however, we can compute it approximately at low cost (within a finite region in $\mathbb{Z}^2$) based on the approach introduced in [22].

A 2D explicit continuous curve is represented by

$$E = \left\{(x, y) \in \mathbb{R}^2 : y = f(x)\right\}, \tag{3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. From (2), for E in (3), $D_S(E)$ can be written as

$$D_S(E) = \left\{(x_{\text{int}}, y_{\text{int}}) \in \mathbb{Z}^2 : y_{\text{int}} + t_y = f(x_{\text{int}} + t_x) \text{ for } \exists (t_x, t_y) \in S\right\}. \tag{4}$$

Note that $y_{\text{int}} + t_y = f(x_{\text{int}} + t_x)$ means $(x_{\text{int}} + t_x, y_{\text{int}} + t_y) \in E$. Since f is continuous, the intermediate-value theorem allows for connected S to rewrite (4) as

$$D_S(E) = \left\{(x_{\text{int}}, y_{\text{int}}) \in \mathbb{Z}^2 : \begin{array}{l} y_{\text{int}} \geq \min\limits_{(t_x, t_y) \in S} \left(f(x_{\text{int}} + t_x) - t_y\right), \\ y_{\text{int}} \leq \max\limits_{(t_x, t_y) \in S} \left(f(x_{\text{int}} + t_x) - t_y\right) \end{array}\right\}. \tag{5}$$

Note that both B_∞ and B_1 are connected.

For $S = B_\infty, B_1$, unfortunately, evaluating the minimum and maximum of $f(x_{\text{int}} + t_x) - t_y$ with respect to $(t_x, t_y) \in S$ is practically impossible, because S has infinite elements. Following [22], however, we can approximately compute (5) by replacing $S = B_\infty, B_1$ with finite subsets V_∞, V_1 defined by

$$\begin{aligned} V_\infty &= \left\{\left(-\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}, \\ V_1 &= \left\{\left(-\frac{1}{2}, 0\right), \left(0, -\frac{1}{2}\right), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right)\right\}. \end{aligned}$$

They are the sets of the vertices respectively of B_∞ and B_1 as in Fig. 6. We then obtain the *analytical approximations* for $D_{B_\infty}(E)$ and $D_{B_1}(E)$ respectively as

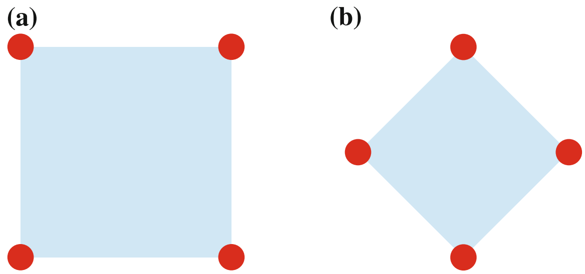

Fig. 6 **a** V_∞ and **b** V_1 (*red points*). They are the sets of the vertices respectively of B_∞ and B_1 (depicted in *blue*)

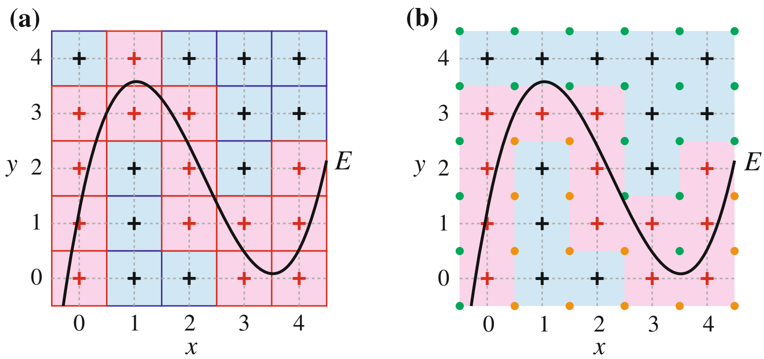

Fig. 7 **a** $D_{B_\infty}(E)$ and **b** $D'_{V_\infty}(E)$ (*red points* on the grids) for $E = \{(x, y) \in \mathbb{R}^2 : y = f(x) = 0.4571x^3 - 3.127x^2 + 5.019x + 1.228\}$. In (**b**), points $(x, y) \in \mathbb{Z}^2 \oplus V_\infty$ satisfying $y > f(x)$ are depicted in *green*, while those satisfying $y < f(x)$ in *orange*; an integer point $\boldsymbol{v} \in \mathbb{Z}^2$ is in $D'_{V_\infty}(E)$ iff $\{\boldsymbol{v}\} \oplus V_\infty$ (four points) are depicted in both colors or include a point on E

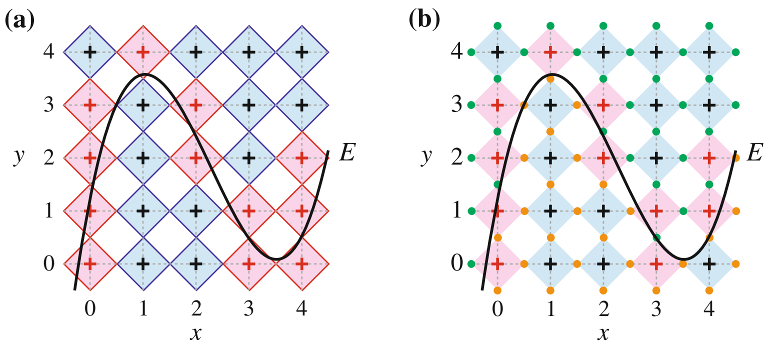

Fig. 8 **a** $D_{B_1}(E)$ and **b** $D'_{V_1}(E)$ (*red points* on the grids) for $E = \{(x, y) \in \mathbb{R}^2 : y = f(x) = 0.4571x^3 - 3.127x^2 + 5.019x + 1.228\}$. In **b**, points $(x, y) \in \mathbb{Z}^2 \oplus V_1$ satisfying $y > f(x)$ are depicted in *green*, while those satisfying $y < f(x)$ in *orange*; an integer point $\boldsymbol{v} \in \mathbb{Z}^2$ is in $D'_{V_1}(E)$ iff $\{\boldsymbol{v}\} \oplus V_1$ (four points) are depicted in both colors or include a point on E

$$D'_{V_\infty}(E) = \left\{ (x_{\text{int}}, y_{\text{int}}) \in \mathbb{Z}^2 : \begin{array}{l} y_{\text{int}} \geq \min\left\{ f\left(x_{\text{int}} - \frac{1}{2}\right), f\left(x_{\text{int}} + \frac{1}{2}\right) \right\} - \frac{1}{2}, \\ y_{\text{int}} \leq \max\left\{ f\left(x_{\text{int}} - \frac{1}{2}\right), f\left(x_{\text{int}} + \frac{1}{2}\right) \right\} + \frac{1}{2} \end{array} \right\}, \tag{6}$$

$$D'_{V_1}(E) = \left\{ (x_{\text{int}}, y_{\text{int}}) \in \mathbb{Z}^2 : \begin{array}{l} y_{\text{int}} \geq \min\left\{ f\left(x_{\text{int}} - \frac{1}{2}\right), f\left(x_{\text{int}} + \frac{1}{2}\right), f\left(x_{\text{int}}\right) - \frac{1}{2} \right\}, \\ y_{\text{int}} \leq \max\left\{ f\left(x_{\text{int}} - \frac{1}{2}\right), f\left(x_{\text{int}} + \frac{1}{2}\right), f\left(x_{\text{int}}\right) + \frac{1}{2} \right\} \end{array} \right\}. \tag{7}$$

For each $(x_{\text{int}}, y_{\text{int}}) \in \mathbb{Z}^2$, the inequalities in (5) are evaluated at only the four vertices of $S = B_\infty$ (resp. B_1) in $D'_{V_\infty}(E)$ (resp. $D'_{V_1}(E)$), while they have to be evaluated at all the points in $S = B_\infty$ (resp. B_1) in the morphological discretization. Therefore the analytical approximation is computationally more inexpensive than the morphological discretization. On the other hand, the analytical approximation may fail in collecting some integer points involved in the morphological discretization as in Figs. 7 and 8 (in the next section, we will see that such cases arise when the Minkowski addition of an integer point and B_∞ [resp. B_1] is intersected by E, but not by its piecewise linear approximation defined in (8) [resp. (9)]). We remark that we can also replace B_∞ and B_1 with larger finite subsets than V_∞ and V_1 for more accurate approximation.

3 Connectivity Relation Between Morphological Discretization and Analytical Approximation

In this section, we show that the analytical approximation for a 2D explicit curve introduced in the last section has the same connectivity in $\mathbb{Z}^2$ that the morphological discretization has. To prove this, we show that the discretization of an explicit curve E by the analytical approximation can be seen as the morphological discretization of a piecewise linear approximation of E. We first show that $D'_{V_\infty}(E)$ has the same connectivity with $D_{B_\infty}(E)$.

Theorem 2 *$D'_{V_\infty}(E)$ is 1-connected.*

Proof We show that $D'_{V_\infty}(E) = D_{B_\infty}(E')$ for E' defined by

$$E' = \{(x, y) \in \mathbb{R}^2 : y = f'(x)\}, \tag{8}$$

where

$$\begin{aligned} f'(x) = f&\left(\left\lfloor x + \tfrac{1}{2}\right\rfloor - \tfrac{1}{2}\right) \\ &+ \left(x - \left(\left\lfloor x + \tfrac{1}{2}\right\rfloor - \tfrac{1}{2}\right)\right)\left(f\left(\left\lfloor x + \tfrac{1}{2}\right\rfloor + \tfrac{1}{2}\right) - f\left(\left\lfloor x + \tfrac{1}{2}\right\rfloor - \tfrac{1}{2}\right)\right). \end{aligned}$$

E' is a piecewise linear approximation of E as in Fig. 9. We remark that $f'(x_{\mathrm{int}} + \frac{1}{2}) = f(x_{\mathrm{int}} + \frac{1}{2})$, and $f'(x)$ is linear within $\left[x_{\mathrm{int}} - \frac{1}{2}, x_{\mathrm{int}} + \frac{1}{2}\right]$ for $\forall x_{\mathrm{int}} \in \mathbb{Z}$. $D_{B_\infty}(E')$ is 1-connected because E' is connected in $\mathbb{R}^2$.

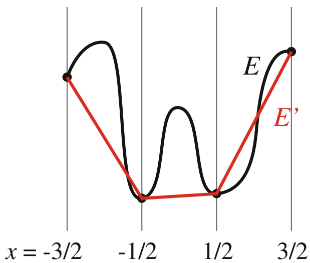

Fig. 9 E and E'

From (5), $D_{B_\infty}(E')$ is written as

$$D_{B_\infty}(E') = \left\{ (x_{\text{int}}, y_{\text{int}}) \in \mathbb{Z}^2 : \begin{array}{l} y_{\text{int}} \geq \min\limits_{(t_x, t_y) \in B_\infty} \left(f'(x_{\text{int}} + t_x) - t_y \right), \\ y_{\text{int}} \leq \max\limits_{(t_x, t_y) \in B_\infty} \left(f'(x_{\text{int}} + t_x) - t_y \right) \end{array} \right\}.$$

We will transform it into $D'_{V_\infty}(E)$. Since t_x and t_y are independent of each other for $(t_x, t_y) \in B_\infty$, the extrema of $f'(x_{\text{int}} + t_x) - t_y$ with respect to (t_x, t_y) are obtained by minimizing and maximizing it independently for t_x and t_y. With respect to t_y, it is obviously minimal with $t_y = \frac{1}{2}$, and maximal with $t_y = -\frac{1}{2}$. With respect to t_x, on the other hand, the extrema are at $t_x = -\frac{1}{2}$ or $\frac{1}{2}$ because $f'(x)$ is linear within $\left[x_{\text{int}} - \frac{1}{2}, x_{\text{int}} + \frac{1}{2}\right]$ for $\forall x_{\text{int}} \in \mathbb{Z}$. Consequently,

$$D_{B_\infty}(E') = \left\{ (x_{\text{int}}, y_{\text{int}}) \in \mathbb{Z}^2 : \begin{array}{l} y_{\text{int}} \geq \min \left\{ f'\left(x_{\text{int}} - \frac{1}{2}\right), f'\left(x_{\text{int}} + \frac{1}{2}\right) \right\} - \frac{1}{2}, \\ y_{\text{int}} \leq \max \left\{ f'\left(x_{\text{int}} - \frac{1}{2}\right), f'\left(x_{\text{int}} + \frac{1}{2}\right) \right\} + \frac{1}{2} \end{array} \right\},$$

which is equal to $D'_{V_\infty}(E)$ in (6), because $f'(x_{\text{int}} \pm \frac{1}{2}) = f(x_{\text{int}} \pm \frac{1}{2})$ for $\forall x_{\text{int}} \in \mathbb{Z}$. □

We next show that $D'_{V_1}(E)$ has the same connectivity with $D_{B_1}(E)$.

Theorem 3 *$D'_{V_1}(E)$ is 0-connected.*

Proof We show that $D'_{V_1}(E) = D_{B_1}(E'')$ for E'' defined by

$$E'' = \{(x, y) \in \mathbb{R}^2 : y = f''(x)\}, \tag{9}$$

where

$$f''(x) = f\left(\tfrac{\lfloor 2x \rfloor}{2}\right) + 2\left(x - \tfrac{\lfloor 2x \rfloor}{2}\right)\left(f\left(\tfrac{\lfloor 2x \rfloor + 1}{2}\right) - f\left(\tfrac{\lfloor 2x \rfloor}{2}\right)\right).$$

E'' is a piecewise linear approximation of E as in Fig. 10. We remark that $f''(\frac{x_{\text{int}}}{2}) = f(\frac{x_{\text{int}}}{2})$, and $f''(x)$ is linear within $\left[\frac{x_{\text{int}}}{2}, \frac{x_{\text{int}}+1}{2}\right]$ for $\forall x_{\text{int}} \in \mathbb{Z}$. From Theorem 1, $D_{B_1}(E'')$ is 0-connected because E'' is connected in $\mathbb{R}^2$.

$D_{B_1}(E'')$ is written as

$$D_{B_1}(E'') = \left\{ (x_{\text{int}}, y_{\text{int}}) \in \mathbb{Z}^2 : \begin{array}{l} y_{\text{int}} \geq \min\limits_{(t_x, t_y) \in B_1} \left(f''(x_{\text{int}} + t_x) - t_y \right), \\ y_{\text{int}} \leq \max\limits_{(t_x, t_y) \in B_1} \left(f''(x_{\text{int}} + t_x) - t_y \right) \end{array} \right\}.$$

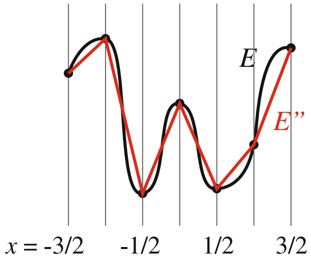

Fig. 10 E and E''

We will transform it into $D'_{V_1}(E)$. Since $-\frac{1}{2} + |t_x| \leq t_y \leq \frac{1}{2} - |t_x|$ for $(t_x, t_y) \in B_1$, $D_{B_1}(E'')$ can be rewritten as

$$D_{B_1}(E'') = \left\{ (x_{\text{int}}, y_{\text{int}}) \in \mathbb{Z}^2 : \begin{array}{l} y_{\text{int}} \geq \min\limits_{t_x \in \left[-\frac{1}{2}, \frac{1}{2}\right]} \left(f''(x_{\text{int}} + t_x) - \dfrac{1}{2} + |t_x| \right), \\ y_{\text{int}} \leq \max\limits_{t_x \in \left[-\frac{1}{2}, \frac{1}{2}\right]} \left(f''(x_{\text{int}} + t_x) + \dfrac{1}{2} - |t_x| \right) \end{array} \right\}.$$

Here, the minimum of $f''(x_{\text{int}} + t_x) - \frac{1}{2} + |t_x|$ and the maximum of $f''(x_{\text{int}} + t_x) + \frac{1}{2} - |t_x|$ with respect to $t_x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ are at $t_x = -\frac{1}{2}$, 0 or $\frac{1}{2}$, because they are linear within $t_x \in \left[-\frac{1}{2}, 0\right]$ and $t_x \in \left[0, \frac{1}{2}\right]$. We thus obtain

$$D_{B_1}(E'') = \left\{ (x_{\text{int}}, y_{\text{int}}) \in \mathbb{Z}^2 : \begin{array}{l} y_{\text{int}} \geq \min \left\{ f''\left(x_{\text{int}} - \dfrac{1}{2}\right), f''\left(x_{\text{int}} + \dfrac{1}{2}\right), f''(x_{\text{int}}) - \dfrac{1}{2} \right\}, \\ y_{\text{int}} \leq \max \left\{ f''\left(x_{\text{int}} - \dfrac{1}{2}\right), f''\left(x_{\text{int}} + \dfrac{1}{2}\right), f''(x_{\text{int}}) + \dfrac{1}{2} \right\} \end{array} \right\},$$

which is equal to $D'_{V_1}(E)$ in (7), because $f''(x_{\text{int}} \pm \frac{1}{2}) = f(x_{\text{int}} \pm \frac{1}{2})$, and $f''(x_{\text{int}}) = f(x_{\text{int}})$ for $\forall x_{\text{int}} \in \mathbb{Z}$. $\square$

For 2D explicit curves, therefore, the analytical approximation introduced in [22] guarantees the same connectivity in $\mathbb{Z}^2$ that the morphological discretization does.

4 Conclusion

We investigated the connectivity relation between the morphological discretization and the analytical approximation introduced in [22] for 2D explicit continuous curves. We first showed that the morphological discretization of a 2D continuous curve with the structuring element B_1 (the ball of radius $\frac{1}{2}$ based on l_1 norm) guarantees 0-

connectivity of the obtained result. We then showed that the discretization of a 2D explicit curve by the analytical approximation has the same connectivity in $\mathbb{Z}^2$ that its morphological discretization has. Our proof is based on the idea that the analytical approximation for a 2D explicit curve can be seen as the morphological discretization of a piecewise linear approximation of the curve. Whether this property holds for parametric curves, or curves and surfaces in higher dimensions, will be investigated in our future work.

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