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Optimal Consensus Set and Preimage of 4-Connected Circles in a Noisy Environment

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Abstract

This paper exploits the problem of fitting special forms of annuli that correspond to 4-connected digital circles to a given set of points in 2D images in the presence of noise by maximizing the number of inliers, namely the consensus set. We prove that the optimal solutions can be described by solutions with three points on the annulus boundary. These solutions correspond to vertices of the preimage of the annulus in the parameter space thus allowing us to build the preimage and to enumerate all the optimal solutions.

1. Introduction

There are many facets to pattern recognition. They include the problem of effectively recognizing primitives such as lines, circles or planes in the case of man-made objects, in a noisy environment. This problem can be formulated as a fitting problem of noisy data to a given primitive and an optimal solution recognizes the primitive. We can find many different methods for this problem in image analysis as well as in the more general setting of function or surface fitting/dimension reduction [4, 5, 6, 7]. These methods have reasonably low computational time complexity in exchange for approximate/sub-optimal solutions.

Besides reducing time complexity, enumerating all the optimal solutions is an important aspect in recognizing primitives. Having a way of finding all the optimal solutions gives a way to choose among those optimal solutions if need be and more importantly, provides a way to evaluate more efficient (in term of time complexity) methods that propose sub-optimal solutions. In this paper, we exploit the problem of finding (all) the optimal solutions in fitting a circle to 2D noisy data.

We have shown in a previous paper that most, if not all, types of commonly used digital circles can be described analytically as an annulus [2]. These analytical descriptions are based on morphological type digitization schemes. In this paper we use Adjacency 0-Flakes as structuring elements [3]. This defines 4-connected circles. The method described in this paper works exactly in the same way for various other types of circles such as 8-connected circles, disks, etc. The problem to solve is finding the parameters of all the primitive annuli that contain the most inliers (reciprocally the least outliers). Our aim here is the search for optimal solutions and possibly the whole multidimensional set of the optimal solutions. We prove that if an optimal solution exists then there exists another optimal solution with a limited number of points (three for a circle) that lie on the boundary of the annulus. Once this is established, firstly it gives a direct method to compute all the optimal solutions with points on the boundary. Secondly, it is easy to see that those solutions represent characteristic points of the optimal solution region in the parameter space. Indeed, when an optimal solution has a point on its boundary, a small change in the parameters will make it an outlier.

2 Digitization using adjacency flakes

After a short recall on notations and basic definitions, we are going to present the digitization scheme we are considering in this paper.

Let $\{e_1, \ldots, e_n\}$ denote the canonical basis of the

n-dimensional Euclidean vector space. Let $\mathbb{Z}^n$ be the subset of $\mathbb{R}^n$ that consists of all the integer coordinate points. A *digital (resp. Euclidean) point* is an element of $\mathbb{Z}^n$ (resp. $\mathbb{R}^n$). We denote by x_i the i-th coordinate of a point or a vector x, that is its coordinate associated to e_i. A *digital (resp. Euclidean) object* is a set of digital (resp. Euclidean) points. For all $k \in \{0, \ldots, n-1\}$, two digital points v and w are said to be *k-adjacent* or *k-neighbors*, if for all $i \in \{1, \ldots, n\}$, $|v_i - w_i| \leq 1$ and $\sum_{j=1}^{n} |v_j - w_j| \leq n - k$. In the 2-dimensional plane, the 0- and 1-neighborhood notations correspond respectively to the classical 8- and 4-neighborhood notations.

2.1 Digitization

The digitization scheme we are considering in this paper is an *Adjacency Flake Digitization* [3]. It is based on a morphological based digitization scheme with a structuring element called an *Adjacency Flake*. We are going, in this paper, to consider the 2D 0-adjacency flake (or simply 0-Flake) mostly but the reasoning goes as well for a 2D 1-adjacency flake structuring element and other structuring elements that define most, if not all, classical digital circle types [2]. Let us define these digitization schemes.

Let $\oplus$ be the Minkowski addition, known as dilation, such that $\mathcal{A} \oplus \mathcal{B} = \cup_{b \in \mathcal{B}} \{a + b : a \in \mathcal{A}\}$. The dilation of a Euclidean primitive by a structuring element defines an *offset zone*. All the digital points in the offset zone form the digital object. The offset zone can, in many cases, be analytically characterized which allows a global analytical description of the digital primitive.

$$\mathcal{D}_{\mathcal{A}}(\mathcal{S}) = (\mathcal{S} \oplus \mathcal{A}) \cap \mathbb{Z}^n,$$

where $\mathcal{D}_{\mathcal{A}}(\mathcal{S})$ is the digitization of the Euclidean object $\mathcal{S}$, and $\mathcal{A}$ is a structuring element.

A simple way of defining a structuring element is to consider unit balls for a given distance. The naive digitization model is defined that way with the Manhattan distance $d_1 = ||.||_1$. The supercover and the standard model definitions are based on the Chebishev distance $d_\infty = ||.||_\infty$ (see Figure 1 (top)) [1]. These norms can be regrouped in a set of norms called the k-adjacency norms:

$$[.]_k = \max\left\{||.||_\infty, \frac{||x||_1}{n-k}\right\}$$

See Figure 1 (bottom). These norms have been introduced in [3] and used to define digital analytical circles and spheres in [2]. The k-adjacency norms have been named that way because, for two points v and w in $\mathbb{Z}^n$, v and w are k-neighbors iff $[v - w]_k \leq 1$.

A k-adjacency flake (or k-Flake) is a subset of the k-adjacency norm ball defined as follows:

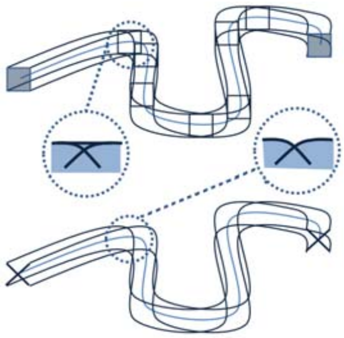


Figure 1. Difference between digitization offsets with (top) the Chebishev half-unit ball and (bottom) the F_0 flake.

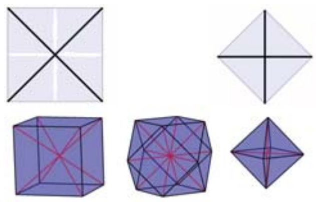


Figure 2. The k-Flakes in 2D (F_0,F_1) and 3D (F_0,F_1,F_2).

Definition 1. *The k-adjacency flakes (or k-Flake) [2], F_k, based on the k-adjacency norm $[.]_k$ unit ball (of radius $1/2$) is defined as a set of straight line segments joining the vertices of the ball and crossing the center of the ball.*

The Figure 2 shows 2D- and 3D-Flakes.

The 0-Flake digitization of a hypersphere $S = \mathcal{S}(c, r)$ of center c, radius r is analytically described by:

$$D_{F_0}(S) = \left\{ x \in \mathbb{Z}^n : \begin{array}{c} -\sum_{i=1}^{n} |x_i - c_i| - \frac{1}{2} \leq \\ \sum_{i=1}^{n} (x_i - c_i)^2 - r^2 \\ \leq \sum_{i=1}^{n} |x_i - c_i| - \frac{1}{2} \\ \text{for } r \geq \sqrt{n}/2. \end{array} \right\}$$

The smallest possible 0-Flake hypersphere is of radius $\sqrt{n}/2$. With a flake structuring element, hypersphere of smaller radii are not correctly defined. This is one of the limitations of the flake model. It is however not a big constraint as it corresponds to a circle that spans only a couple of voxels.

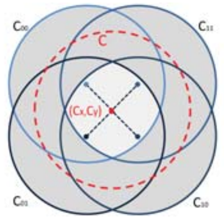


Figure 3. 0**-Flake digitization of a circle.**

2.2 0-Flake digital circle

In this paper, we focus on the digitization of circles using the flake model F_0 [2]. The Figure 3 shows the 0-Flake offset (in grey) of a circle (in doted red). We call **border circles** the 4 circles that form the border of the annulus, i.e. the circles centered on $(c_1 \pm \frac{1}{2}, c_2 \pm \frac{1}{2})$. In Figure 3, we can see the four border circles C_{00}, C_{01}, C_{10} and C_{11} (in blue). The border circles are noted C_{ij} where i and j encode the translation of the center of the border circle from the center of the Euclidean circle (c_1, c_2) we want to digitize. The generalization leads to the border circle definition:

Definition 2. *Let $\mathcal{C}_{ij}$ be a border circle of the flake annulus $\mathcal{C}(c_1, c_2)$ of radius R. The equation of $\mathcal{C}_{ij}$ is:* $\left(x - \left(c_1 + \frac{(-1)^i}{2}\right)\right)^2 + \left(y - \left(c_2 + \frac{(-1)^j}{2}\right)\right)^2 = R^2.$

3 Three point boundary theorem

We show now that for each flake annulus there exists an equivalent flake annulus with three points on its external boundary (defining the preimage vertices for the optimal solutions):

Theorem 1. *Let $S = \{p_i\}_{i\in[1,m]}$ be a set of points in $\mathbb{R}^2$ such that the maximal distance between two points of the set is greater than the size of two pixels (if it is not the case than the optimal circle is obviously the 0-Flake circle of radius $\sqrt{2}/2$). Let $C(c_1, c_2)$ of radius R be a 0-Flake circle and thickness 1 such that $\forall k \in [1, m], p_k \in C$. Then there exists another flake circle $C'(c'_1, c'_2)$ of radius R' also containing all the points and such that three among them are on one of the border circles of C.*

Let S be a set of m ($m \geq 3$) points in $\mathbb{R}^2$. Let C be the 0-Flake annulus of center $C(C_x, C_y)$, radius R and width 1 such that C covers S. Our first assumption is that no point of S is on the annulus boundary.

The theorem proof is given in three steps:

Proof:

1 A translation of C allows to reach the first point P_1. We just have to translate in direction of the inlier that is closest to the boundary.

2 A radius reduction while keeping P_1 on the boundary allows to reach a second point P_2. By continuously reducing the radius, the offset zone decreases. Either we obtain a second point on the boundary or we obtain a radius of $\sqrt{2}/2$.

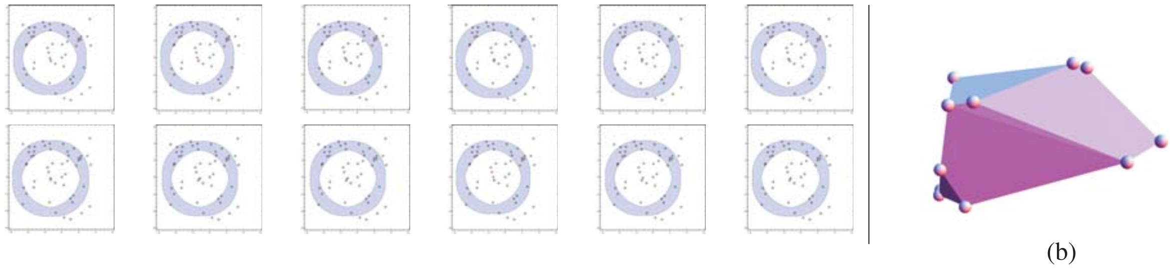
3 For the third point, the continuous transformation of the flake circle that leads to it, is a little trickier. Once you have two points on the border circles, in order to keep them on the border circles, the center of the flake circle has only one degree of freedom left. The possible center positions correspond to a straight line L. The radius of course varies also when moving the center along L.

When the center moves to infinity on either side of the straight line L, the radius itself becomes infinite. The boundary of the flake circle becomes two straight parallel lines. One of them is the straight line $P - 1 - P_2$ and the other one is parallel to $P_1 - P_2$ and one or the other side depending on which side of L the center is. Now, two configurations can appear. The first is that the distance between P_1 and P_2 is greater or equal to $\sqrt{2}$. In this case, the center can move along the whole line L and it is easy to see that a third point will at some time be reached. The worst case being that all the points are aligned with P_1 and P_2 in which case we have an infinite radius. If the distance between P_1 and P_2 is smaller than $\sqrt{2}$ then there is a segment on the line $P_1 - P_2$ where the center can not be located because we would have a negative radius. However, before this, when continuously moving the center towards the points P_1 and P_2, the radius becomes $\sqrt{2}/2$. If we have no third point on the boundary at that time, we have our optimal radius flake circle. □

The following theorem states that we can build a finite number of digital 0-Flake circles with 3 points on its boundary.

Theorem 2. *There are 64 0-Flake circles that have 3 given non-aligned points on its border circles.*

Proof. A flake annulus represents the surface formed by 4 border circles (Figure 3); this means that given three points P_1, P_2 and P_3, we must locate them on one or many of the fours circles. There exists $4^3 = 64$ different possibilities for the three points. Every configuration among the 64 defines one unique flake circle. □



(a) (b)

Figure 4. a) Fitting of 2D noisy 0-Flake circles. b) Preimage.

4 Finding the optimal fitting flake annuli

Let us consider a set S of m points in $\mathbb{R}^2$ (or $\mathbb{Z}^n$). Let us now propose a fitting method that finds the 0-Flake circles that maximizes the number of inliers. The principle of the fitting algorithm, based on Theorem 1 and Theorem 2, is rather straightforward. For every three points of the set of points S, we compute all the possible flake annuli and count the number of inliers. The optimal annuli are the ones that enclose the maximum number of inliers. The output is a set $\mathbf{V}$ of parameter values $(c_1^{op}, c_2^{op}, R^{op})$ corresponding to the fitted flake annuli that give the optimal consensus sets. The time complexity of the algorithm is thus $O(m^4)$ for m points in S.

4.1 Experiments

We used Mathematica for implementing our method. We applied our method for 2D noisy flake annuli as shown in Figure 4. In this example we have 14 optimal consensus sets with three points on the annuli border. This proves that our method is capable of detecting mutiple optimal consensus sets. As we can see from Theorems 1 and 2, each of these 14 solutions corresponds to a vertex of the preimage (optimal solution region in the parameter space) of the optimal consensus set. This allows to construct the region containing all the possible optimal solutions in the parameter space (Figure 4.b).

5 Conclusion and perspectives

In this paper we have presented a new method for fitting flake annuli to a set of points with a width of 1. Our approach guarantees optimal results. To the best of our knowledge, this is the first work that yields all the consensus sets for flake circles.

One of the future work concerns fitting of 3D flake annuli. For the 3D fitting the same algorithm is used however 4 points are needed instead of three in order to define the flake annulus. Our proposed method allows to construct the preimage of the optimal solutions. We are now considering alternative methods of constructing this polytope with a smaller time complexity. The optimal solution polytope in the parameter space is in some sense the kernel of the solutions to the fitting problem. Considering solutions that are not optimal but almost optimal leads to other polytopes. Each of these polytopes encompassing the others as the constraints are relaxed. This should allow us a new insight into the solution spaces for noisy fitting problems. A last perspective is of course fitting of other type of curves such as conics for instance.

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